

## CH 10.5: ANOVA

- $(ab - 1)S^2$  : total sum of squares.
- Let  $X_{ij} \sim N(\mu_{ij}, \sigma^2)$ ,  $1 \leq i \leq a$ ,  $1 \leq j \leq b$  be indep. with

$$\mu_{ij} = \mu + \alpha_i + \beta_j$$

satisfying

$$\sum \alpha_i = 0, \quad \sum \beta_j = 0.$$

$$H_0 : \left. \begin{array}{l} \mu_{11} = \cdots = \mu_{1b} \\ \mu_{21} = \cdots = \mu_{2b} \\ \cdots \cdots \cdots \cdots \cdots \cdots \\ \mu_{a1} = \cdots = \mu_{ab} \end{array} \right\} \iff$$

$$\beta_1 = \cdots = \beta_b = 0.$$

Home Page

Title Page

◀ ▶

◀ ▶

Page 2 of 100

Go Back

Full Screen

Close

Quit

$$H_0 : \left. \begin{array}{l} \mu_{11} = \cdots = \mu_{a1} \\ \mu_{12} = \cdots = \mu_{a2} \\ \cdots \cdots \cdots \cdots \cdots \cdots \\ \mu_{1b} = \cdots = \mu_{ab} \end{array} \right\} \iff$$

$$\alpha_1 = \cdots = \alpha_a = 0.$$

Home Page

Title Page

◀ ▶

◀ ▶

Page 3 of 100

Go Back

Full Screen

Close

Quit

- Test  $H_0$  use LRT.

$$H_0 : \beta_1 = \cdots = \beta_b = 0$$

$$\Theta = \left\{ \begin{aligned} &\mu_{ij} = \mu + \alpha_i + \beta_j, \\ &\sum \alpha_i = 0, \sum \beta_j = 0 \end{aligned} \right\}$$

$$\Theta_0 = \left\{ \mu_{ij} = \mu + \alpha_i, \sum \alpha_i = 0 \right\}.$$

$$L(\hat{\Theta}) =$$

$$\left( \frac{ab/(2\pi)}{\sum_{i,j} (X_{ij} + \bar{X}_{..} - \bar{X}_{i.} - \bar{X}_{.j})^2} \right)^{-\frac{ab}{2}} \cdot e^{-\frac{ab}{2}}.$$

$$L(\hat{\Theta}_0) = \left( \frac{ab/(2\pi)}{\sum \sum (X_{ij} - \bar{X}_{i.})^2} \right)^{-\frac{ab}{2}} e^{-\frac{ab}{2}}.$$

Home Page

Title Page

◀ ▶

◀ ▶

Page 5 of 100

Go Back

Full Screen

Close

Quit

$$\lambda = \frac{L(\hat{\Theta}_0)}{L(\hat{\Theta})}$$
$$= \left( \frac{\sum \sum (X_{ij} - \overline{X}_{ij} - \overline{X}_{.j} + \overline{X}_{..})^2}{\sum \sum (X_{ij} - \overline{X}_{i.})^2} \right)^{\frac{ab}{2}}.$$

Home Page

Title Page

◀ ▶

◀ ▶

Page 6 of 100

Go Back

Full Screen

Close

Quit

$$\begin{aligned}\lambda^{\frac{ab}{2}} &= \frac{Q_5}{Q_5 + \sum_i \sum_j (\overline{X_{.j}} - \overline{X_{..}})^2} \\ &= \frac{1}{1 + \frac{\sum \sum (\overline{X_{.j}} - \overline{X_{..}})^2}{Q_5}} \\ &= \frac{1}{1 + \frac{Q_4}{Q_5}}.\end{aligned}$$

Home Page

Title Page

◀ ▶

◀ ▶

Page 7 of 100

Go Back

Full Screen

Close

Quit

Then

$$(ab - 1)S^2 = Q_2 + Q_4 + Q_5.$$

$$F = \frac{Q_4/(b - 1)}{Q_5/(a - 1)(b - 1)}$$
$$\sim^{H_0} F(b - 1, (a - 1)(b - 1))$$

with critical value

$$c = F_\alpha(b - 1, (a - 1)(b - 1)).$$



When  $H_1$  is true, the noncentrality parameters of  $Q_4$  and  $Q_5$  are:

$$\frac{\sum_{j=1}^b (\mu_{.j} - \mu)^2}{\sigma^2} = \frac{a \sum_{j=1}^b \beta_j^2}{\sigma^2},$$

$$\frac{\sum \sum (\mu_{ij} - \mu_{i.} - \mu_{.j} + \mu)^2}{\sigma^2} = 0.$$

Home Page

Title Page

◀ ▶

◀ ▶

Page 9 of 100

Go Back

Full Screen

Close

Quit

Test:

$$H_0 : \alpha_1 = \cdots = \alpha_a = 0$$

use

$$F = \frac{Q_2/(a-1)}{Q_5/(a-1)(b-1)} \geq c.$$

Home Page

Title Page

◀ ▶

◀ ▶

Page 10 of 100

Go Back

Full Screen

Close

Quit

- Let  $(X_i, Y_i)$  with linear model:

$$Y_i = \alpha + \beta(X_i - \bar{X}) + \varepsilon_i$$

with  $\varepsilon_i \sim N(0, \sigma^2)$ .

$$L(\alpha, \beta, \sigma^2) = \prod_{i=1}^N \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{(Y_i - \alpha - \beta(X_i - \bar{X}))^2}{2\sigma^2} \right\}.$$

Home Page

Title Page

◀ ▶

◀ ▶

Page 11 of 100

Go Back

Full Screen

Close

Quit

MLE:

$$\begin{aligned}\hat{\beta} &= \frac{\sum (Y_i - \bar{Y})(X_i - \bar{X})}{\sum (X_i - \bar{X})^2} \\ &= \frac{\sum (X_i - \bar{X})Y_i}{\sum (X_i - \bar{X})^2}.\end{aligned}$$

$$\begin{cases} \hat{\alpha} = \bar{Y}, \\ \hat{\sigma}^2 = \frac{1}{n} \sum \left[ Y_i - \hat{\alpha} - \hat{\beta}(X_i - \bar{X}) \right]^2. \end{cases}$$

$$E(\hat{\alpha}) = \alpha,$$

$$Var(\hat{\alpha}) = \frac{1}{n}\sigma^2,$$

$$E(\hat{\beta}) = \frac{\sum(X_i - \bar{X})E(Y_i)}{\sum(X_i - \bar{X})^2} = \beta,$$

$$Var(\hat{\beta}) = \sum \left( \frac{X_i - \bar{X}}{\sum(X_i - \bar{X})^2} \right)^2 \sigma^2$$
$$\cdot Var(Y_i) = \frac{\sigma^2}{\sum(X_i - \bar{X})^2}.$$

Home Page

Title Page

◀ ▶

◀ ▶

Page 13 of 100

Go Back

Full Screen

Close

Quit

$$\sum_{i=1}^n (Y_i - \alpha - \beta(X_i - \bar{X}))^2 =$$

$$n(\hat{\alpha} - \alpha)^2 + (\hat{\beta} - \beta)^2 \sum (X_i - \bar{X})^2$$

$$+ n\hat{\sigma}^2,$$

i.e.

$$Q/\sigma^2 = Q_1/\sigma^2 + Q_2/\sigma^2 + Q_3/\sigma^2$$

$$\chi^2(n) \quad \chi^2(1) \quad \chi^2(1) \quad \chi^2(n-2)$$

Home Page

Title Page

◀ ▶

◀ ▶

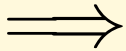
Page 14 of 100

Go Back

Full Screen

Close

Quit



$$\begin{aligned} T_1 &= \frac{\sqrt{n}(\hat{\alpha} - \alpha)/\sigma}{\sqrt{Q_3/(\sigma^2(n-2))}} \\ &= \frac{\hat{\alpha} - \alpha}{\sqrt{\sigma^2/(n-2)}} \\ &\sim t(n-2). \end{aligned}$$

$$\begin{aligned} T_2 &= \frac{\sqrt{\sum (X_i - \bar{X})^2 (\hat{\beta} - \beta) / \sigma}}{\sqrt{Q_3 / \sigma^2 (n - 2)}} \\ &= \frac{\hat{\beta} - \beta}{\sqrt{n \hat{\sigma}^2 / ((n - 2) \sum (X_i - \bar{X})^2)}} \\ &\sim t(n - 2). \end{aligned}$$

Home Page

Title Page

◀ ▶

◀ ▶

Page 16 of 100

Go Back

Full Screen

Close

Quit



## Ch10.7 A test of Independence

- Let  $(X_i, Y_i)$  iid.  $\sim$

$$N \left( \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right).$$

$$H_0 : \rho = 0, \quad H_1 : \rho \neq 0.$$

- $\lambda = f(|R|)$  where

$$R = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum (X_i - \bar{X}) \sum (Y_i - \bar{Y})^2}}.$$

- $\lambda \leq \lambda_0 \Leftrightarrow |R| \geq c$  reject  $H_0$
- Let  $R_c$  be the correlation coefficient given  $X_i = x_i$

$$\frac{R_c \sqrt{\sum (Y_i - \bar{Y})^2}}{\sum (X_i - \bar{X})^2} = \frac{\sum (X_i - \bar{X}) Y_i}{\sum (X_i - \bar{X})^2}.$$

We have

$$\frac{R_c \sqrt{n-2}}{\sqrt{1-R_c}} \sim t(n-2).$$

$\Rightarrow$

$$R \sim f(r) =$$

$$\frac{\Gamma((n-1)/2)}{\Gamma(\frac{1}{2})\Gamma((n-2)/2)} (1-r^2)^{\frac{n-4}{2}}, |r| < 1$$

Home Page

Title Page

◀ ▶

◀ ▶

Page 19 of 100

Go Back

Full Screen

Close

Quit

and

$$P(|R| > c_1) = P(|T| \geq c_2).$$

• Let

$$W = \frac{1}{2} \log\left(\frac{1+R}{1-R}\right) \sim N\left(\frac{1}{2} \log\left(\frac{1+\rho}{1-\rho}\right), \frac{1}{n-3}\right).$$



Home Page

Title Page



Page 20 of 100

Go Back

Full Screen

Close

Quit

Use

$$Z = \frac{W - \frac{1}{2} \log[(1 + \rho_0)/(1 - \rho_0)]}{\sqrt{1/(n - 3)}} \\ \sim^{H_0} N(0, 1)$$

to test  $H_0 : \rho = \rho_0$ .

Home Page

Title Page

◀ ▶

◀ ▶

Page 21 of 100

Go Back

Full Screen

Close

Quit

# Thank all of you!

Home Page

Title Page

◀▶

◀▶

Page 22 of 100

Go Back

Full Screen

Close

Quit