CH 10.5: ANOVA

- $(ab-1)S^2$: total sum of squares.

- Let $X_{ij} \sim N(\mu_{ij}, \sigma^2)$, $1 \le i \le a, \quad 1 \le j \le b$ be indep. with

$$\mu_{ij} = \mu + \alpha_i + \beta_j$$

satisfying

$$\sum \alpha_i = 0, \quad \sum \beta_j = 0.$$

Home Page
Title Page
Page 1 of 100
Go Back
Full Screen
Close
Quit

$$
H_0: \left.\begin{array}{c} \mu_{11} = \cdots = \mu_{1b} \\ \mu_{21} = \cdots = \mu_{2b} \\ \cdots\cdots\cdots\cdots \\ \mu_{a1} = \cdots = \mu_{ab} \end{array}\right\} \Longleftrightarrow
$$

$$
\beta_1 = \cdots = \beta_b = 0.
$$

Home Page

Title Page

Go Back

Full Screen

Close

Quit

$$H_0 : \left.\begin{array}{c} \mu_{11} = \cdots = \mu_{a1} \\ \mu_{12} = \cdots = \mu_{a2} \\ \cdots\cdots\cdots\cdots \\ \mu_{1b} = \cdots = \mu_{ab} \end{array}\right\} \Longleftrightarrow$$

$$\alpha_1 = \cdots = \alpha_a = 0.$$

- Test H_0 use LRT.

$$H_0 : \beta_1 = \cdots = \beta_b = 0$$

$$\Theta = \{\mu_{ij} = \mu + \alpha_i + \beta_j, \sum \alpha_i = 0, \sum \beta_j = 0\}$$

$$\Theta_0 = \{\mu_{ij} = \mu + \alpha_i, \sum \alpha_i = 0\}.$$

$$L(\hat{\Theta}) =$$

$$\left(\frac{ab/(2\pi)}{\sum_{i,j}(X_{ij} + \overline{X}_{..} - \overline{X}_{i.} - \overline{X}_{.j})^2}\right)^{-\frac{ab}{2}}$$

$$\cdot e^{-\frac{ab}{2}}.$$

$$L(\hat{\Theta}_0) = (\frac{ab/(2\pi)}{\sum\sum(X_{ij} - \overline{X}_{i.})^2})^{-\frac{ab}{2}} e^{-\frac{ab}{2}}.$$

$$\lambda = \frac{L(\hat{\Theta}_0)}{L(\hat{\Theta})}$$

$$= \left(\frac{\sum\sum(X_{ij} - \overline{X_{ij}} - \overline{X_{.j}} + \overline{X_{..}})^2}{\sum\sum(X_{ij} - \overline{X_{i.}})^2}\right)^{\frac{ab}{2}}.$$

$$\lambda^{\frac{ab}{2}} = \frac{Q_5}{Q_5 + \sum_i \sum_j (\overline{X_{.j}} - \overline{X_{..}})^2}$$
$$= \frac{1}{1 + \frac{\sum\sum(\overline{X_{.j}} - \overline{X_{..}})^2}{Q_5}}$$
$$= \frac{1}{1 + \frac{Q_4}{Q_5}}.$$

Then

$$(ab-1)S^2 = Q_2 + Q_4 + Q_5.$$

$$F = \frac{Q_4/(b-1)}{Q_5/(a-1)(b-1)}$$

$$\sim^{H_0} F(b-1,(a-1)(b-1))$$

with critical value

$$c = F_\alpha(b-1,(a-1)(b-1)).$$

When H_1 is true, the noncentrality parameters of Q_4 and Q_5 are:

$$\frac{\sum_{j=1}^{b}(\mu_{,j} - \mu)^2}{\sigma^2} = \frac{a\sum_{j=1}^{b}\beta_j^2}{\sigma^2},$$

$$\frac{\sum\sum(\mu_{ij} - \mu_{i.} - \mu_{.j} + \mu)^2}{\sigma^2} = 0.$$

Test:

$$H_0 : \alpha_1 = \cdots = \alpha_a = 0$$

use

$$F = \frac{Q_2/(a-1)}{Q_5/(a-1)(b-1)} \geq c.$$

- Let (X_i, Y_i) with liner model:

$$Y_i = \alpha + \beta(X_i - \overline{X}) + \varepsilon_i$$

with $\varepsilon_i \sim N(0, \sigma^2)$.

$$L(\alpha, \beta, \sigma^2) = \prod_{i=1}^{N} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\{ -\frac{(Y_i - \alpha - \beta(X_i - \overline{X}))^2}{2\sigma^2} \}.$$

MLE:

$$\hat{\beta} = \frac{\sum (Y_i - \overline{Y})(X_i - \overline{X})}{\sum (X_i - \overline{X})^2} = \frac{\sum (X_i - \overline{X}) Y_i}{\sum (X_i - \overline{X})^2}.$$

$$\begin{cases} \hat{\alpha} = \overline{Y}, \\ \hat{\sigma}^2 = \frac{1}{n} \sum \left[Y_i - \hat{\alpha} - \hat{\beta}(X_i - \overline{X}) \right]^2. \end{cases}$$

$$E(\hat{\alpha}) = \alpha,$$

$$Var(\hat{\alpha}) = \frac{1}{n}\sigma^2,$$

$$E(\hat{\beta}) = \frac{\sum(X_i - \overline{X})E(Y_i)}{\sum(X_i - \overline{X})^2} = \beta,$$

$$Var(\hat{\beta}) = \sum\left(\frac{X_i - \overline{X}}{\sum(X_i - \overline{X})^2}\right)^2$$

$$\cdot \ Var(Y_i) = \frac{\sigma^2}{\sum(X_i - \overline{X})^2}.$$

$$\sum_{i=1}^{n}(Y_i - \alpha - \beta(X_i - \overline{X}))^2 =$$

$$n(\hat{\alpha} - \alpha)^2 + (\hat{\beta} - \beta)^2 \sum (X_i - \overline{X})^2$$

$$+ n\hat{\sigma}^2,$$

i.e.

$$Q/\sigma^2 = Q_1/\sigma^2 + Q_2/\sigma^2 + Q_3/\sigma^2$$

$$\chi^2(n) \qquad \chi^2(1) \quad \chi^2(1) \quad \chi^2(n-2)$$

$\Longrightarrow$

$$T_1 = \frac{\sqrt{n}(\hat{\alpha} - \alpha)/\sigma}{\sqrt{Q_3/(\sigma^2(n-2))}}$$

$$= \frac{\hat{\alpha} - \alpha}{\sqrt{\sigma^2/(n-2)}}$$

$$\sim t(n-2).$$

$$T_2 = \frac{\sqrt{\sum(X_i - \overline{X})^2}(\hat{\beta} - \beta)/\sigma}{\sqrt{Q_3/\sigma^2(n-2)}}$$

$$= \frac{\hat{\beta} - \beta}{\sqrt{n\hat{\sigma}^2/((n-2)\sum(X_i - \overline{X})^2)}}$$

$$\sim t(n-2).$$

Ch10.7 A test of Independence

- Let (X_i, Y_i) iid. $\sim$

$$N\left(\begin{pmatrix}\mu_1\\ \mu_2\end{pmatrix}, \begin{pmatrix}\sigma_1^2 & \rho\sigma_1\sigma_2\\ \rho\sigma_1\sigma_2 & \sigma_2^2\end{pmatrix}\right).$$

$$H_0 : \rho = 0, \qquad H_1 : \rho \neq 0.$$

- $\lambda = f(|R|)$ where

$$R = \frac{\sum (X_i - \bar{X})(Y_i - \bar{Y})}{\sqrt{\sum (X_i - \bar{X}) \sum (Y_i - \bar{Y})^2}}.$$

- $\lambda \leq \lambda_0 \leftrightarrow |R| \geq c$ reject H_0

- Let R_c be the correlation coefficient given $X_i = x_i$

$$\frac{R_c \sqrt{\sum (Y_i - \bar{Y})^2}}{\sum (X_i - \bar{X})^2} = \frac{\sum (X_i - \bar{X}) Y_i}{\sum (X_i - \bar{X})^2}.$$

We have

$$\frac{R_c\sqrt{n-2}}{\sqrt{1-R_c}} \sim t(n-2).$$

$\Longrightarrow$

$$R \sim f(r) =$$

$$\frac{\Gamma((n-1)/2)}{\Gamma(\frac{1}{2})\Gamma((n-2)/2)}(1-r^2)^{\frac{n-4}{2}}, |r| < 1$$

and

$$P(|R| > c_1) = P(|T| \geq c_2).$$

- Let

$$W = \frac{1}{2}\log(\frac{1+R}{1-R}) \sim N\Big(\frac{1}{2}\log(\frac{1+\rho}{1-\rho}), \frac{1}{n-3}\Big).$$

Use

$$Z = \frac{W - \frac{1}{2}\log[(1+\rho_0)/(1-\rho_0)]}{\sqrt{1/(n-3)}}$$

$$\sim^{H_0} N(0,1)$$

to test $H_0 : \rho = \rho_0$.

Thank all of you!