Ch 10. Inferences About Normal Models

Ch10.1: The Distri. of Quadratic Forms

- Quadratic Forms: $X \in R^n$, $A \in \mathfrak{M}_{n\times n}$

$$X'AX + b'X + c,$$

where $b \in R^n, c \in R^1$.

Example:

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$$
$$= \frac{1}{n-1}\sum_{i=1}^{n}(e_i'X - \frac{1}{n}\mathbf{1}'X)^2$$
$$= \frac{1}{n-1}\sum_{i=1}^{n}((e_i - \frac{1}{n}\mathbf{1})'X)^2$$

$$= \frac{1}{n-1}\sum_{i=1}^{n} X'(e_i - \frac{1}{n}\mathbf{1})(e_i - \frac{1}{n}\mathbf{1})'X$$

$$= \frac{1}{n-1}X'(I - \frac{2}{n}\mathbf{1}\mathbf{1}' + \frac{1}{n}\mathbf{1}\mathbf{1}')X$$

$$= \frac{1}{n-1}X'(I - \frac{1}{n}\mathbf{1}\mathbf{1}')X.$$

- If $X \sim N(\mu\mathbf{1}, \sigma^2 I_n)$, then

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1).$$

- Let

$$Q = \sum_{i=1}^{k} Q_i,$$

where Q_i be quadratic forms of Normal distribution X with $X \sim N(\mu, \sigma^2 I_n)$.

Theorem 1: If $\frac{Q}{\sigma^2}, \frac{Q_1}{\sigma^2}, \cdots, \frac{Q_{k-1}}{\sigma^2}$ have $\chi^2(r), \chi^2(r_1), \cdots, \chi^2(r_{k-1})$, and $Q_k \geq 0$. Then

(1). $Q_1, \cdots, Q_k$ are independent.

(2). $\frac{Q_k}{\sigma^2} \sim \chi^2(r - (r_1 + \cdots + r_{k-1})) = \chi^2(r_k)$.

Let $X \sim N(\mu, \sigma^2)$, a, b are positive integers, $n = ab$.

Samples:

$$\begin{cases} X_{11} \cdots X_{1j} \cdots X_{1b} & \bar{X}_{1.} \\ X_{21} \cdots X_{2j} \cdots X_{2b} & \bar{X}_{2.} \\ \cdots \cdots \cdots \cdots \cdots & \\ X_{a1} \cdots X_{aj} \cdots X_{ab} & \overline{X}_{a.} \end{cases}$$

$$
\begin{cases}
\bar{X} = \dfrac{1}{ab}\displaystyle\sum_{i}\sum_{j} X_{ij} \\
\bar{X}_{i.} = \dfrac{1}{b}\displaystyle\sum_{j=1}^{b} X_{ij} \\
\overline{X_{.j}} = \dfrac{1}{a}\displaystyle\sum_{i=1}^{a} X_{ij}
\end{cases}
$$

Example:

$$
\begin{aligned}
&(ab-1)S^2 \\
&= \sum_i \sum_j (X_{ij} - \overline{X_{..}})^2 \\
&= \sum_i \sum_j (X_{ij} - \overline{X_{i.}} + \overline{X_{i.}} - \overline{X_{..}})^2
\end{aligned}
$$

$$= \sum_i \sum_j (X_{ij} - \overline{X_{i.}})^2$$

$$+ b \sum_{i=1}^{a} (\overline{X_{i.}} - \overline{X_{..}})$$

$$\Longrightarrow$$

$$Q = Q_1 + Q_2.$$

$$\frac{Q}{\sigma^2} \sim \chi^2(ab-1),$$

$$\frac{Q_1}{\sigma^2} = \sum_{i=1} \frac{\sum_{j=1}^{b}(X_{ij}-\overline{X_{i.}})^2}{\sigma^2}.$$

$\frac{\sum_{j=1}^{b}(X_{ij}-\overline{X_{i.}})^2}{\sigma^2} \sim \chi^2(b-1)$ are independent for i.

$$\Rightarrow \frac{Q_1}{\sigma^2} \sim \chi^2(a(b-1)), \quad Q_2 \geq 0,$$

$$\Rightarrow \frac{Q_2}{\sigma^2} \sim \chi^2(ab-1-a(b-1))$$
$$= \chi^2(a-1)$$

and $\frac{Q_1}{\sigma^2}$ are independent.

Example 2 : In $(ab-1)S^2$, use

$$X_{ij}-\overline{X_{..}}=X_{ij}-\overline{X_{.j}}+\overline{X_{.j}}-\overline{X_{..}}$$

$$\begin{aligned} Q &= \sum\sum(X_{ij}-\overline{X_{.j}})^2 \\ &\quad + a\sum_{j=1}^{b}(\overline{X_{.j}}-\overline{X_{..}})^2 \\ &= Q_3+Q_4, \end{aligned}$$

$Q_3 \sim \chi^2(b(a-1)),\ Q_4 \sim \chi^2(b-1).$

Example 3 : In $(ab-1)S^2$, use

$$X_{ij} - \overline{X}_{..} = (X_{i.} - \overline{X}_{..}) + (\overline{X}_{.j} - \overline{X}_{..}) + X_{ij} - \overline{X}_{i.} - \overline{X}_{.j} + \overline{X}_{..},$$

$$(ab-1)S^2 = b\sum_i (X_{i.} - \overline{X}_{..})^2 + a\sum_j (\overline{X}_{.j} - \overline{X}_{..})^2 + \sum_i \sum_j (X_{ij} - \overline{X}_{i.} - \overline{X}_{.j} + \overline{X}_{..})^2.$$

$$= Q_2 + Q_4 + Q_5,$$

$$Q_5 \geq 0, \quad \frac{Q}{\sigma^2} \sim \chi^2(ab-1),$$

$$\frac{Q_2}{\sigma^2} \sim \chi^2(a-1), \quad \frac{Q_4}{\sigma^2} \sim \chi^2(b-1),$$

$\Rightarrow$

$$\frac{Q_5}{\sigma^2} \sim \chi^2((a-1)(b-1)),$$

Q_2, Q_4 and Q_5 are independent.

$\Rightarrow$

$$\frac{Q_4/\sigma^2(b-1)}{Q_3/\sigma^2(b(a-1))} = \frac{Q_4/(b-1)}{Q_3/(b(a-1))},$$

$$\frac{Q_4/\sigma^2(b-1)}{Q_5/\sigma^2((b-1)(a-1))} = \frac{Q_4/(b-1)}{Q_5/((b-1)(a-1))}.$$

Ch 10.2. A Test of the Equality of Several Means

- Let $X_{ij}, \cdots, X_{aj}$ iid. $\sim N(\mu, \sigma^2)$,$1 \leq j \leq b$.

Test:

$$H_0 : \mu_1 = \mu_2 = \cdots = \mu_b = \mu$$

$$H_1 : \mu_l \neq \mu_k, \text{for some } l, k, \text{ and } l \neq k.$$

$$\Theta = \{(\mu_1, \mu_2, \cdots, \mu_b, \sigma^2);$$
$$\mu_j \in R^1, \sigma^2 > 0\}$$

$$\Theta_0 = \{(\mu_1, \mu_2, \cdots, \mu_b, \sigma^2);$$
$$\mu_1 = \mu_2 = \cdots = \mu_b \in R^1, \sigma^2 > 0\}.$$

$$L(\Theta) = (\frac{1}{2\pi\sigma^2})^{\frac{ab}{2}} \exp\{ -\frac{1}{2\sigma^2} \sum_i \sum_j (X_{ij} - \mu)^2 \}$$

$$L(\Theta_0) = (\frac{1}{2\pi\sigma^2})^{\frac{ab}{2}} \exp\{ -\frac{1}{2\sigma^2} \sum_j \sum_i (X_{ij} - \mu_j)^2 \}.$$

$\Rightarrow$

$$L(\hat{\Theta}_0) = \left(\frac{ab}{2\pi \sum_i \sum_j (X_{ij} - \overline{X})^2}\right)^{\frac{ab}{2}} \cdot e^{-\frac{ab}{2}}$$

$$L(\hat{\Theta}) = \left(\frac{ab}{2\pi \sum_j \sum_i (X_{ij} - \overline{X}_{.j})^2}\right)^{\frac{ab}{2}} \cdot e^{-\frac{ab}{2}}$$

$$\Rightarrow \lambda = \frac{\hat{\hat{\Theta}}_0}{\hat{\hat{\Theta}}}$$

$$= \left(\frac{\sum_j \sum_i (X_{ij} - \overline{X}_{,j})^2}{\sum_j \sum_i (X_{ij} - \overline{X}_{..})^2} \right)^{\frac{ab}{2}}$$

$$\lambda^{\frac{2}{ab}} = \frac{Q_3}{Q} = \frac{Q_3}{Q_3 + Q_4} = \frac{1}{1 + \frac{Q_4}{Q_3}}.$$

If $\lambda \leq \lambda_0$, **reject** H_0.

Given

$$\alpha = P_{H_0}\left[\frac{1}{1+\frac{Q_4}{Q_3}} \le \lambda_0^{\frac{2}{ab}}\right]$$

$$= P_{H_0}\left[\frac{Q_4/(b-1)}{Q_3/(b(a-1))} \ge c\right],$$

then

$$c = \frac{b(a-1)}{b-1}\left(\lambda_0^{-\frac{2}{ab}} - 1\right)$$

$$= F_\alpha(b-1, b(a-1)).$$

If the power of the test $H_0 \longleftrightarrow H_1$ when H_0 is false

$$\frac{Q_4}{\sigma^2} \sim \chi^2(b-1)?$$

Ch10.3: Noncentral χ^2 and Noncentral F

- Let $X_1, \cdots, X_n$ be independent with $X_i \sim N(\mu_i, \sigma^2), i = 1, 2, \cdots, n.$

$$Y = \sum_{i=1}^{n} X_i/\sigma^2 \sim \chi^2(n, \theta)$$

with $\theta = \frac{\sum_{i=1}^{n} \mu_i^2}{\sigma^2}.$

- Normal quadratic form

$$Q(X_1, \cdots, X_n)\sigma^2 \sim \chi^2(r, \theta)$$

with $\mu = Q(\mu_1, \cdots, \mu_n)/\sigma^2$.

- If $Q(X_1, \cdots, X_n)/\sigma^2 \sim \chi^2(r, \theta)$ for certain $(\mu_{10}, \cdots, \mu_{n0})$, then it is for all $(\mu_1, \cdots, \mu_n)$.

- If $U \sim \chi^2(r_1, \theta), \quad V \sim \chi^2(r_2)$ are independent, then

$$\frac{U/r_1}{V/r_2} \sim F(r_1, r_2; \theta).$$

Density of $\chi_n^2(\delta)$

$$\chi_n^2(x,\delta) = e^{-\frac{\delta^2}{2}} \sum_{i=0} c_{in} \frac{(\delta^2/2)^i}{i!} x^{\frac{n}{2}+i-1} e^{-\frac{x}{2}}$$

for $x > 0$, where $c_{in} = [2^{\frac{n}{2}+i}\Gamma(\frac{n}{2}+i)]^{-1}$.

Density of $t_n^2(\delta)$

$$
\begin{aligned}
& t_n(x,\delta) \\
= \; & e^{-\frac{\delta^2}{2}}(n+x^2)^{-\frac{n+1}{2}} \sum_{i=0} c_{in} \frac{(\delta x)^i}{i!} \left(\frac{2}{n+x^2}\right)^{\frac{i}{2}},
\end{aligned}
$$

where $c_{in} = \dfrac{n^{\frac{n}{2}}\Gamma(\frac{n+i+1}{2})}{\sqrt{n}\Gamma(\frac{n}{2})}$.

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Density of $F_{m,n}(\delta)$

$$f_{m,n}(x,\delta)$$
$$= e^{-\frac{\delta^2}{2}} \sum_{i=0} c_{imn} \frac{\left(\frac{\delta^2}{2}\right)^i}{i!} \frac{x^{\frac{m}{2}+i-1}}{(n+mx)^{\frac{m+n}{2}+i}}$$

for $x > 0$, where $c_{imn} = \dfrac{n^{\frac{n}{2}} m^{\frac{m}{2}+i} \Gamma(\frac{m+n}{2}+i)}{\Gamma(\frac{n}{2})\Gamma(\frac{m}{2}+i)}$.

- Errors-in-variables linear model:

$$Y_i = \alpha + x_i * \beta + \epsilon_i, \quad X_i = x_i + u_i$$

for $1 \le i \le n$.

- Minimize square orthogonal distance method:

$$\sum_{i=1}^{n} \frac{(Y_i - \alpha - X_i\beta)^2}{1+\beta^2} = \min$$

$$\hat{\alpha} = \bar{Y} - \bar{X}\hat{\beta},$$

$$\hat{\beta} = \frac{-(s_{xx}^2 - s_{yy}^2) + \sqrt{(s_{xx}^2 - s_{yy}^2)^2 + 4 * s_{xy}^4}}{2s_{xy}^2},$$

where

$$s_{xx}^2 = \frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2,$$

$$s_{yy}^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2.$$