

# Ch 10. Inferences About Normal Models

## Ch10.1: The Distri. of Quadratic Forms

- Quadratic Forms:  $X \in R^n$ ,  $A \in \mathfrak{M}_{n \times n}$

$$X'AX + b'X + c,$$

where  $b \in R^n$ ,  $c \in R^1$ .

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## Example:

$$\begin{aligned} S^2 &= \frac{1}{n-1} \sum_{i=1}^n (X_i - \bar{X})^2 \\ &= \frac{1}{n-1} \sum_{i=1}^n \left( e_i' X - \frac{1}{n} \mathbf{1}' X \right)^2 \\ &= \frac{1}{n-1} \sum_{i=1}^n \left( \left( e_i - \frac{1}{n} \mathbf{1} \right)' X \right)^2 \end{aligned}$$

$$= \frac{1}{n-1} \sum_{i=1}^n X'(e_i - \frac{1}{n}\mathbf{1})(e_i - \frac{1}{n}\mathbf{1})'X$$

$$= \frac{1}{n-1} X'(I - \frac{2}{n}\mathbf{1}\mathbf{1}' + \frac{1}{n}\mathbf{1}\mathbf{1}')X$$

$$= \frac{1}{n-1} X'(I - \frac{1}{n}\mathbf{1}\mathbf{1}')X.$$

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- If  $X \sim N(\mu\mathbf{1}, \sigma^2 I_n)$ , then

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi^2(n-1).$$

- Let

$$Q = \sum_{i=1}^k Q_i,$$

where  $Q_i$  be quadratic forms of Normal distribution  $X$  with  $X \sim N(\mu, \sigma^2 I_n)$ .

**Theorem 1:** If  $\frac{Q}{\sigma^2}, \frac{Q_1}{\sigma^2}, \dots, \frac{Q_{k-1}}{\sigma^2}$  have  $\chi^2(r), \chi^2(r_1), \dots, \chi^2(r_{k-1})$ , and  $Q_k \geq 0$ .

Then

(1).  $Q_1, \dots, Q_k$  are independent.

(2).  $\frac{Q_k}{\sigma^2} \sim \chi^2(r - (r_1 + \dots + r_{k-1})) = \chi^2(r_k)$ .

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Let  $X \sim N(\mu, \sigma^2)$ ,  $a, b$  are positive integers,  $n = ab$ .

Samples:

$$\left\{ \begin{array}{lllll} X_{11} & \cdots & X_{1j} & \cdots & X_{1b} & \bar{X}_1. \\ X_{21} & \cdots & X_{2j} & \cdots & X_{2b} & \bar{X}_2. \\ \cdots & \cdots & \cdots & \cdots & \cdots & \\ X_{a1} & \cdots & X_{aj} & \cdots & X_{ab} & \bar{X}_a. \end{array} \right.$$

$$\left\{ \begin{array}{l} \bar{X} = \frac{1}{ab} \sum_i \sum_j X_{ij} \\ \bar{X}_{i.} = \frac{1}{b} \sum_{j=1}^b X_{ij} \\ \overline{X}_{.j} = \frac{1}{a} \sum_{i=1}^a X_{ij} \end{array} \right.$$

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Example:

$$\begin{aligned} & (ab - 1)S^2 \\ = & \sum_i \sum_j (X_{ij} - \overline{X_{..}})^2 \\ = & \sum_i \sum_j (X_{ij} - \overline{X_{i.}} + \overline{X_{i.}} - \overline{X_{..}})^2 \end{aligned}$$

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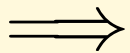
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$$= \sum_i \sum_j (X_{ij} - \overline{X_{i.}})^2$$
$$+ b \sum_{i=1}^a (\overline{X_{i.}} - \overline{X_{..}})^2$$



$$Q = Q_1 + Q_2.$$

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$$\frac{Q}{\sigma^2} \sim \chi^2(ab - 1),$$

$$\frac{Q_1}{\sigma^2} = \sum_{i=1} \frac{\sum_{j=1}^b (X_{ij} - \overline{X_{i.}})^2}{\sigma^2}.$$

$$\frac{\sum_{j=1}^b (X_{ij} - \overline{X_{i.}})^2}{\sigma^2} \sim \chi^2(b - 1) \text{ are independent}$$

for  $i$ .

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$$\Rightarrow \frac{Q_1}{\sigma^2} \sim \chi^2(a(b-1)), \quad Q_2 \geq 0,$$

$$\begin{aligned} \Rightarrow \frac{Q_2}{\sigma^2} &\sim \chi^2(ab-1-a(b-1)) \\ &= \chi^2(a-1) \end{aligned}$$

and  $\frac{Q_1}{\sigma^2}$  are independent.

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**Example 2** : In  $(ab - 1)S^2$ , use

$$X_{ij} - \overline{X}_{..} = X_{ij} - \overline{X}_{.j} + \overline{X}_{.j} - \overline{X}_{..}$$

$$\begin{aligned} Q &= \sum \sum (X_{ij} - \overline{X}_{.j})^2 \\ &\quad + a \sum_{j=1}^b (\overline{X}_{.j} - \overline{X}_{..})^2 \\ &= Q_3 + Q_4, \end{aligned}$$

$$Q_3 \sim \chi^2(b(a - 1)), \quad Q_4 \sim \chi^2(b - 1).$$

Example 3 : In  $(ab - 1)S^2$ , use

$$\begin{aligned} X_{ij} - \overline{X_{..}} &= (X_{i.} - \overline{X_{..}}) + (\overline{X_{.j}} - \overline{X_{..}}) \\ &\quad + X_{ij} - \overline{X_{i.}} - \overline{X_{.j}} + \overline{X_{..}}, \end{aligned}$$

$$\begin{aligned} (ab - 1)S^2 &= b \sum_i (X_{i.} - \overline{X_{..}})^2 \\ &\quad + a \sum_j (\overline{X_{.j}} - \overline{X_{..}})^2 \\ &\quad + \sum_i \sum_j (X_{ij} - \overline{X_{i.}} - \overline{X_{.j}} + \overline{X_{..}})^2. \end{aligned}$$

$$= Q_2 + Q_4 + Q_5,$$

$$Q_5 \geq 0, \quad \frac{Q_5}{\sigma^2} \sim \chi^2(ab - 1),$$

$$\frac{Q_2}{\sigma^2} \sim \chi^2(a - 1), \quad \frac{Q_4}{\sigma^2} \sim \chi^2(b - 1),$$

$\Rightarrow$

$$\frac{Q_5}{\sigma^2} \sim \chi^2((a - 1)(b - 1)),$$

$Q_2, Q_4$  and  $Q_5$  are independent.

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$\Rightarrow$

$$\frac{Q_4/\sigma^2(b-1)}{Q_3/\sigma^2(b(a-1))} = \frac{Q_4/(b-1)}{Q_3/(b(a-1))},$$

$$\begin{aligned}
 & \frac{Q_4/\sigma^2(b-1)}{Q_5/\sigma^2((b-1)(a-1))} \\
 = & \frac{Q_4/(b-1)}{Q_5/((b-1)(a-1))}.
 \end{aligned}$$

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# Ch 10.2. A Test of the Equality of Several Means

- Let  $X_{1j}, \dots, X_{aj}$  iid.  $\sim N(\mu, \sigma^2), 1 \leq j \leq b$ .

Test:

$$H_0 : \mu_1 = \mu_2 = \dots = \mu_b = \mu$$

$$H_1 : \mu_l \neq \mu_k, \text{ for some } l, k, \\ \text{and } l \neq k.$$



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$$\Theta = \{(\mu_1, \mu_2, \dots, \mu_b, \sigma^2);$$
$$\mu_j \in R^1, \sigma^2 > 0\}$$

$$\Theta_0 = \{(\mu_1, \mu_2, \dots, \mu_b, \sigma^2);$$
$$\mu_1 = \mu_2 = \dots = \mu_b \in R^1, \sigma^2 > 0\}.$$

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$$L(\Theta) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{ab}{2}} \exp\left\{-\frac{1}{2\sigma^2} \sum_i \sum_j (X_{ij} - \mu)^2\right\}$$

$$L(\Theta_0) = \left(\frac{1}{2\pi\sigma^2}\right)^{\frac{ab}{2}} \exp\left\{-\frac{1}{2\sigma^2} \sum_j \sum_i (X_{ij} - \mu_j)^2\right\}.$$

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⇒

$$L(\hat{\Theta}_0) = \left( \frac{ab}{2\pi \sum_i \sum_j (X_{ij} - \bar{X})^2} \right)^{\frac{ab}{2}} \cdot e^{-\frac{ab}{2}}$$

$$L(\hat{\Theta}) = \left( \frac{ab}{2\pi \sum_j \sum_i (X_{ij} - \bar{X}_{.j})^2} \right)^{\frac{ab}{2}} \cdot e^{-\frac{ab}{2}}$$

$$\Rightarrow \lambda = \frac{\hat{\Theta}_0}{\hat{\Theta}}$$

$$= \left( \frac{\sum_j \sum_i (X_{ij} - \overline{X}_{.,j})^2}{\sum_j \sum_i (X_{ij} - \overline{X}_{..})^2} \right)^{\frac{ab}{2}}$$

$$\lambda_{\frac{2}{ab}} = \frac{Q_3}{Q} = \frac{Q_3}{Q_3 + Q_4} = \frac{1}{1 + \frac{Q_4}{Q_3}}.$$

If  $\lambda \leq \lambda_0$ , reject  $H_0$ .

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Given

$$\begin{aligned}\alpha &= P_{H_0} \left[ \frac{1}{1 + \frac{Q_4}{Q_3}} \leq \lambda_0^{\frac{2}{ab}} \right] \\ &= P_{H_0} \left[ \frac{Q_4/(b-1)}{Q_3/(b(a-1))} \geq c \right],\end{aligned}$$

then

$$\begin{aligned}c &= \frac{b(a-1)}{b-1} \left( \lambda_0^{-\frac{2}{ab}} - 1 \right) \\ &= F_\alpha(b-1, b(a-1)).\end{aligned}$$

If the power of the test  $H_0 \longleftrightarrow H_1$  when  $H_0$  is false

$$\frac{Q_4}{\sigma^2} \sim \chi^2(b - 1)?$$

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## Ch10.3: Noncentral $\chi^2$ and Noncentral F

- Let  $X_1, \dots, X_n$  be independent with  $X_i \sim N(\mu_i, \sigma^2)$ ,  $i = 1, 2, \dots, n$ .

$$Y = \sum_{i=1}^n X_i / \sigma^2 \sim \chi^2(n, \theta)$$

with  $\theta = \frac{\sum_{i=1}^n \mu_i^2}{\sigma^2}$ .

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- Normal quadratic form

$$Q(X_1, \dots, X_n)\sigma^2 \sim \chi^2(r, \theta)$$

with  $\mu = Q(\mu_1, \dots, \mu_n)/\sigma^2$ .

- If  $Q(X_1, \dots, X_n)/\sigma^2 \sim \chi^2(r, \theta)$  for certain  $(\mu_{10}, \dots, \mu_{n0})$ , then it is for all  $(\mu_1, \dots, \mu_n)$ .



• If  $U \sim \chi^2(r_1, \theta)$ ,  $V \sim \chi^2(r_2)$  are independent, then

$$\frac{U/r_1}{V/r_2} \sim F(r_1, r_2; \theta).$$

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## Density of $\chi_n^2(\delta)$

$$\chi_n^2(x, \delta) = e^{-\frac{\delta^2}{2}} \sum_{i=0}^{\infty} c_{in} \frac{(\delta^2/2)^i}{i!} x^{\frac{n}{2}+i-1} e^{-\frac{x}{2}}$$

for  $x > 0$ , where  $c_{in} = [2^{\frac{n}{2}+i} \Gamma(\frac{n}{2} + i)]^{-1}$ .

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## Density of $t_n^2(\delta)$

$$t_n(x, \delta) = e^{-\frac{\delta^2}{2}} (n + x^2)^{-\frac{n+1}{2}} \sum_{i=0}^{\infty} c_{in} \frac{(\delta x)^i}{i!} \left( \frac{2}{n + x^2} \right)^{\frac{i}{2}},$$

where  $c_{in} = \frac{n^{\frac{n}{2}} \Gamma(\frac{n+i+1}{2})}{\sqrt{n} \Gamma(\frac{n}{2})}$ .

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## Density of $F_{m,n}(\delta)$

$$f_{m,n}(x, \delta) = e^{-\frac{\delta^2}{2}} \sum_{i=0}^{\infty} c_{imn} \frac{\left(\frac{\delta^2}{2}\right)^i x^{\frac{m}{2}+i-1}}{i! (n + mx)^{\frac{m+n}{2}+i}}$$

for  $x > 0$ , where  $c_{imn} = \frac{n^{\frac{n}{2}} m^{\frac{m}{2}+i} \Gamma(\frac{m+n}{2}+i)}{\Gamma(\frac{n}{2}) \Gamma(\frac{m}{2}+i)}$ .

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- Errors-in-variables linear model:

$$Y_i = \alpha + x_i * \beta + \epsilon_i, \quad X_i = x_i + u_i$$

for  $1 \leq i \leq n$ .

- Minimize square orthogonal distance method:

$$\sum_{i=1}^n \frac{(Y_i - \alpha - X_i\beta)^2}{1 + \beta^2} = \min$$

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$$\hat{\alpha} = \bar{Y} - \bar{X} \hat{\beta},$$

$$\hat{\beta} = \frac{-(s_{xx}^2 - s_{yy}^2) + \sqrt{(s_{xx}^2 - s_{yy}^2)^2 + 4 * s_{xy}^4}}{2s_{xy}^2},$$

where

$$s_{xx}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \bar{X})^2,$$

$$s_{yy}^2 = \frac{1}{n} \sum_{i=1}^n (Y_i - \bar{Y})^2.$$

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