

9.3. LRT

- Let X_1, \dots, X_n are indep. $\sim f_i(x, \theta)$,
 $\theta \in \Theta \subset R^m$, and $\Theta_0 \subset \Theta$.

$$H_0 : \theta \in \Theta_0, \quad H_1 : \theta \in \Theta_1 = \Theta - \Theta_0.$$

$$L(\theta) = \prod_{i=1}^n f_i(x_i; \theta)$$

with $\theta \in \Theta$.

$$\lambda(x_1, \dots, x_n) = \lambda = \frac{\sup_{\theta \in \Theta_0} L(\theta)}{\sup_{\theta \in \Theta} L(\theta)}.$$

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- Likelihood ratio test principle:

$H_0 : \theta \in \Theta_0$ is rejected

$\iff \lambda \leq \lambda_0.$

with

$$\begin{aligned}\alpha &= P\{\lambda(x_1, \dots, x_n) \leq \lambda_0; H_0\} \\ &= \sup_{\theta \in \Theta_0} P_\theta\{\lambda(x_1, \dots, x_n) \leq \lambda_0\}.\end{aligned}$$

Example:

X_1, \dots, X_n iid. $\sim N(\theta_1, \theta_3)$, Y_1, \dots, Y_n
iid. $\sim N(\theta_2, \theta_3)$,

where $\Theta = \{(\theta_1, \theta_2, \theta_3) : \theta_1, \theta_2 \in R^1, \theta_3 > 0\}$

$$H_0 : \theta_1 = \theta_2, \quad H_1 : \theta_1 \neq \theta_2.$$

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$$L(\theta) = \left(\frac{1}{2\pi\theta_3} \right)^{\frac{n+m}{2}} \exp\left\{ -\frac{1}{2\theta_3} \left[\sum_{i=1}^n (x_i - \theta_1)^2 + \sum_{i=1}^m (y_i - \theta_2)^2 \right] \right\}.$$

$$\sup_{\theta \in \Theta_0} L(\theta) = \left(\frac{1}{2\pi e\omega} \right)^{\frac{mn}{2}},$$

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where

$$\omega = \frac{\sum_{i=1}^n (x_i - u)^2 + \sum_{i=1}^m (y_i - u)^2}{m + n},$$

$$u = \frac{n\bar{x} + m\bar{y}}{m + n}.$$

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and

$$\sup_{\theta \in \Theta} L(\theta) = \left(\frac{1}{2\pi e \omega'} \right)^{\frac{mn}{2}}$$

with

$$\omega' = \frac{\sum_{i=1}^n (x_i - \bar{x})^2 + \sum_{i=1}^m (y_i - \bar{y})^2}{m + n}.$$

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Then

$$\begin{aligned}\lambda &= \frac{\sup_{\theta \in \Theta_0} L(\theta)}{\sup_{\theta \in \Theta} L(\theta)} \\ &= \left(\frac{\omega'}{\omega} \right)^{\frac{n+m}{2}},\end{aligned}$$

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where

$$\frac{\omega'}{\omega}$$

$$\begin{aligned} &= \frac{(n-1)S_x^2 + (m-1)S_y^2}{(n-1)S_x^2 + (m-1)S_y^2 + \frac{nm}{m+n}(\bar{X} - \bar{Y})^2} \\ &= \frac{1}{1 + \frac{nm}{m+n} \frac{(\bar{X} - \bar{Y})^2}{(n-1)S_x^2 + (m-1)S_y^2}} \\ &= \frac{m+n-2}{(m+n-2) + T^2}. \end{aligned}$$

with

$$T = \frac{\sqrt{\frac{nm}{m+n}}(\bar{X} - \bar{Y})}{\frac{(n-1)S_x^2 + (m-1)S_y^2}{m+n-2}}$$

• If H_0 holds,

$$T \sim^{H_0} t(n+m-2).$$

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Therefore

$$\lambda \leq \lambda_0 \Leftrightarrow |T| \geq c$$

with $\alpha = P(|T| \geq c; H_0)$.

- $n = 10, m = 6, \alpha = 0.05, \Rightarrow c = 2.145$.

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Def. If W and V are independent, $W \sim N(\delta, 1)$, $V \sim \chi^2(r)$.

$$T = \frac{W}{\sqrt{V/r}}$$

is said to have noncentral t -distribution with r degree of freedom, denote $T \sim t(r; \delta)$.

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If $\theta_1 \neq \theta_2$,

$$\delta = \sqrt{\frac{nm}{n+m}}(\theta_1 - \theta_2) / \sqrt{\theta_3}$$

$$W = \sqrt{\frac{nm}{n+m}}(\bar{X} - \bar{Y}) / \sqrt{\theta_3}$$

$$\sim N(\delta, 1).$$

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$$V = \frac{(n - 1)S_x^2 + (m - 1)S_y^2}{m + n - 2}$$

$$\sim \chi^2(m + n - 2),$$

and

$$T \sim t(r; \delta).$$

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- λ' distribution?

MLE \approx Normal.

$$-2 \ln \lambda \approx^{H_0} \chi^2(r)$$

with

$$r = \dim(\Theta) - \dim(\Theta_0) = 3 - 2 = 1.$$