9.3. LRT

- Let $X_1, \cdots, X_n$ are indep. $\sim f_i(x, \theta)$, $\theta \in \Theta \subset R^m$, and $\Theta_0 \subset \Theta$.

$$H_0 : \theta \in \Theta_0, \quad H_1 : \theta \in \Theta_1 = \Theta - \Theta_0.$$

$$L(\theta) = \prod_{i=1}^{n} f_i(x_i; \theta)$$

with $\theta \in \Theta$.

$$\lambda(x_1, \cdots, x_n) = \lambda = \frac{\sup_{\theta \in \Theta_0} L(\theta)}{\sup_{\theta \in \Theta} L(\theta)}.$$

- Likelihood ratio test principle:

$$H_0 : \theta \in \Theta_0 \text{ is rejected}$$

$$\Longleftrightarrow \lambda \leq \lambda_0.$$

with

$$\alpha = P\{\lambda(x_1, \cdots, x_n) \leq \lambda_0; H_0\}$$
$$= \sup_{\theta \in \Theta_0} P_\theta\{\lambda(x_1, \cdots, x_n) \leq \lambda_0\}.$$

Example:

$X_1, \cdots, X_n$ iid. $\sim N(\theta_1, \theta_3)$, $Y_1, \cdots, Y_n$ iid. $\sim N(\theta_2, \theta_3)$,

where $\Theta = \{(\theta_1, \theta_2, \theta_3) : \theta_1, \theta_2 \in R^1, \theta_3 > 0\}$

$$H_0 : \theta_1 = \theta_2, \qquad H_1 : \theta_1 \neq \theta_2.$$

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$$L(\theta) = \left(\frac{1}{2\pi\theta_3}\right)^{\frac{n+m}{2}} \exp\{-\frac{1}{2\theta_3}$$

$$[\sum^{n}(x_i - \theta_1)^2 + \sum^{m}(y_i - \theta_2)^2]\}.$$

$$\sup_{\theta\in\Theta_0} L(\theta) = \left(\frac{1}{2\pi e\omega}\right)^{\frac{mn}{2}},$$

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where

$$\omega = \frac{\sum_{i=1}^{n}(x_i - u)^2 + \sum_{i=1}^{m}(y_i - u)^2}{m+n},$$

$$u = \frac{n\bar{x} + m\bar{y}}{m+n}.$$

and

$$\sup_{\theta\in\Theta} L(\theta) = \left(\frac{1}{2\pi e\omega'}\right)^{\frac{mn}{2}}$$

with

$$\omega' = \frac{\sum_{i=1}^{n}(x_i - \bar{x})^2 + \sum_{i=1}^{m}(y_i - \bar{y})^2}{m+n}.$$

Then

$$\lambda = \frac{\sup_{\theta\in\Theta_0} L(\theta)}{\sup_{\theta\in\Theta} L(\theta)} = \left(\frac{\omega'}{\omega}\right)^{\frac{n+m}{2}},$$

where

$$\frac{\omega'}{\omega} = \frac{(n-1)S_x^2 + (m-1)S_y^2}{(n-1)S_x^2 + (m-1)S_y^2 + \frac{nm}{m+n}(\bar{X}-\bar{Y})^2}$$

$$= \frac{1}{1 + \frac{nm}{m+n}\frac{(\bar{X}-\bar{Y})^2}{(n-1)S_x^2 + (m-1)S_y^2}}$$

$$= \frac{m+n-2}{(m+n-2)+T^2}.$$

with

$$T = \frac{\sqrt{\frac{nm}{m+n}}(\bar{X} - \bar{Y})}{\frac{(n-1)S_x^2 + (m-1)S_y^2}{m+n-2}}.$$

- If H_0 holds,

$$T \sim^{H_0} t(n+m-2).$$

Therefore

$$\lambda \le \lambda_0 \Leftrightarrow |T| \ge c$$

with $\alpha = P(|T| \ge c; H_0)$.

- $n = 10, m = 6, \alpha = 0.05, \Rightarrow c = 2.145.$

Def. If W and V are independent, $W \sim N(\delta, 1), V \sim \chi^2(r)$.

$$T = \frac{W}{\sqrt{V/r}}$$

is said to have noncentral t-distribution with r degree of freedom, denote $T \sim t(r; \delta)$.

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If $\theta_1 \neq \theta_2$,

$$\delta = \sqrt{\frac{nm}{n+m}}(\theta_1 - \theta_2)/\sqrt{\theta_3}$$

$$W = \sqrt{\frac{nm}{n+m}}(\bar{X} - \bar{Y})/\sqrt{\theta_3}$$

$$\sim N(\delta, 1).$$

$$V = \frac{(n-1)S_x^2 + (m-1)S_y^2}{m+n-2}$$

$$\sim \chi^2(m+n-2),$$

and

$$T \sim t(r;\delta).$$

- λ' distribution?

MLE $\approx$ Normal.

$$-2\ln\lambda \approx^{H_0} \chi^2(r)$$

with

$$r = dim(\Theta) - dim(\Theta_0) = 3 - 2 = 1.$$