

Ch 9. Theory of Statistical Tests

9.1 Certain Best Tests

- How to construct good testing.

For simple hypothesis

$$H_0 : \theta = \theta', \quad H_1 : \theta = \theta'',$$

where $\Theta = \{\theta', \theta''\}$

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1. Define the best test for

$$H_0 \leftrightarrow H_1.$$

2. NP Theorem

3. An example.

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- Use rejection region to define a test.

Given C , if $(x_1, \dots, x_n) \in C$, then reject H_0 , otherwise, accept H_0 .

- The best rejection region $C \iff$ the best test.

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Def : The best rejection region C for the simple hypothesis test of level α :

For any subset A in the sample space satisfying

$$P((X_1, \dots, X_n \in A)) = \alpha :$$

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$$(a). \quad P\{(X_1, \dots, X_n) \in C; H_0\} = \alpha$$

$$(b). \quad P\{(X_1, \dots, X_n) \in C; H_1\} \geq \\ P\{(X_1, \dots, X_n) \in A; H_1\}.$$

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NP Theorem: X_1, \dots, X_n iid. $\sim f(x, \theta)$,

$$L(\theta) = \prod_{i=1}^n f(x_i, \theta),$$

where $\Theta = \{\theta', \theta''\}$. Let

$$C = \left\{ (x_1, \dots, x_n) : \frac{L(\theta')}{L(\theta'')} \leq k \right\}$$

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$$C^* = \left\{ (x_1, \dots, x_n) : \frac{L(\theta')}{L(\theta'')} > k \right\}$$

is a complement of C .

$$\alpha = P \{ (X_1, \dots, X_n) \in C; H_0 \}.$$

Then C is a best rejection region of size α for testing the simple hypothesis $H_0 \leftrightarrow H_1$.

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Proof:

$$\begin{aligned} 0 &\leq? \int_C L(\theta'') - \int_A L(\theta'') \\ &= \int_{C \cap A^*} L(\theta'') - \int_{A \cap C^*} L(\theta'') \\ &\geq \frac{1}{k} \int_{C \cap A^*} L(\theta') - \frac{1}{k} \int_{A \cap C^*} L(\theta') \end{aligned}$$

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$$= \frac{1}{k} \left[\int_C L(\theta') - \int_A L(\theta') \right]$$

$$= \frac{1}{k} (\alpha - \alpha) = 0.$$

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- **Example:** X_1, \dots, X_n iid. $\sim N(\theta, 1)$,
and

$$H_0 : \theta' = 0, H_1 : \theta'' = 1.$$

$$\frac{L(\theta')}{L(\theta'')} = \exp\left\{-\sum_{i=1}^n X_i + \frac{n}{2}\right\} \leq k$$
$$\Leftrightarrow \frac{1}{n} \sum X_i \geq \frac{1}{2} - \frac{1}{n} \ln k =: c$$

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Take

$$C = \{(x_1, \dots, x_n) : \frac{1}{n} \sum_{i=1}^n x_i \geq c\}.$$

If $\alpha = 0.05$, then $c = 1.645/\sqrt{n}$.

$$P(C; H_0) = \alpha,$$

$$\begin{aligned} P(C; H_1) &= P(\bar{X} - 1 \geq c - 1; H_1) \\ &= 1 - \Phi(\sqrt{n}(c - 1)). \end{aligned}$$

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If $n = 25$, then

$$\begin{aligned} P(C; H_1) &= 1 - \Phi(1.645 - 5) \\ &= \Phi(3.355) = 0.999. \end{aligned}$$

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Generalization: H_0 and H_1 are simple hypothesis and X_1, \dots, X_n iid.

$$L_0 = g(x_1, \dots, x_n)$$

is joint pdf. of X_1, \dots, X_n ,

$$L_1 = h(x_1, \dots, x_n)$$

is joint pdf. of X_1, \dots, X_n .

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Then the critical region for $H_0 \leftrightarrow H_1$ is

$$C = \{(x_1, \dots, x_n) : \frac{L_0}{L_1} \leq k\}$$

for some $k > 0$ with

$$\alpha = P((x_1, \dots, x_n) \in C; H_0).$$

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9.2 Uniformly Most Powerful Tests (UMPT)

$$H_0 \leftrightarrow H_1$$

Def.: C is a UMPT of α for the simple hypothesis $H_0 \leftrightarrow$ composite hypothesis $H_1 \Leftrightarrow C$ is a best of α for test $H_0 \leftrightarrow$ each simple hypothesis in H_1 .

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In general UMPT doesn't exist. If it exists,
then NP theorem is used.

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Example 2. Let X_1, \dots, X_n iid. $\sim N(0, \theta)$

$$H_0 : \theta = \theta' \leftrightarrow H_1 : \theta > \theta'$$

where $\Theta = \{\theta : \theta \geq \theta'\}$, the joint pdf. of X_1, \dots, X_n is

$$L(\theta) = \left(\frac{1}{2\pi\theta} \right)^{\frac{n}{2}} \exp\left\{ -\frac{\sum_{i=1}^n x_i^2}{2\theta} \right\}.$$

For any $\theta'' > \theta'$, $k > 0$,

$$\frac{L(\theta')}{L(\theta'')} \leq k$$

$$\Leftrightarrow \left(\frac{\theta''}{\theta'}\right)^{\frac{n}{2}} \exp\left\{-\frac{\theta'' - \theta'}{2\theta'\theta''} \sum x_i^2\right\} \leq k$$

$$\Leftrightarrow \sum x_i^2 \geq \frac{2\theta'\theta''}{\theta'' - \theta'} \left[\frac{n}{2} \ln\left(\frac{\theta''}{\theta'}\right) - \ln k\right]$$

$$=: c.$$

Therefore

$$C = \{(x_1, \dots, x_n) : \sum x_i^2 \geq c\}$$

is the best for

$$H_0 \leftrightarrow H_1 : \theta = \theta'',$$

where c is determined by α .

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$$P\left(\frac{\sum X_i^2}{\theta'} \geq \frac{c_{H_0}}{\theta''}\right) = \alpha$$

$$\Rightarrow c = \theta' \chi_n^2(\alpha).$$

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For $H_1 : \theta = \theta'''$, then

$$C = \{(x_1, \dots, x_n) : \sum x_i^2 \geq c\}$$

is the same and also the best, C is UMPT.

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If $n = 5$, $\alpha = 0.05$, $\theta' = 3$,

$$H_0 : \theta = 3, \quad H_1 : \theta > 3,$$

then

$$c = 3 \times \chi_{15}^2(0.05) = 3 \times 25 = 75.$$

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- Let X_1, \dots, X_n iid. $\sim f(x, \theta)$ ($\theta \in \Theta$).

Suppose that

$$Y = u(x_1, \dots, x_n)$$

is a sufficient, then

$$L(\theta) =$$

$$k_1[u(x_1, \dots, x_n); \theta]k_2(x_1, \dots, x_n),$$

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⇒

$$\begin{aligned}\frac{L(\theta')}{L(\theta'')} &= \frac{k_1(u(x_1, \dots, x_n); \theta')}{k_1(u(x_1, \dots, x_n); \theta'')} \\ &= \frac{k_1(y, \theta')}{k_1(y, \theta'')}.\end{aligned}$$

- A best test or UMPT depends on $u(x_1, \dots, x_n)$ which is sufficient.

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- If $\frac{k_1(y, \theta')}{k_1(y, \theta'')}$ is an increasing function of y for $\theta'' < \theta'$, then

$$\frac{L(\theta')}{L(\theta'')}$$

is called **monotone likelihood ratio** in the statistic $Y = u(x_1, \dots, x_n)$.

Example. X_1, \dots, X_n iid. $\sim N(\theta, 1)$, if

$$\theta'' < \theta',$$

$$\begin{aligned} \frac{L(\theta')}{L(\theta'')} &= \frac{\prod_{i=1}^n \exp\left\{-\frac{1}{2}(x_i - \theta')^2\right\}}{\prod_{i=1}^n \exp\left\{-\frac{1}{2}(x_i - \theta'')^2\right\}} \\ &= \exp\left\{(\theta' - \theta'') \sum x_i - \frac{n}{2}\theta'^2\right. \\ &\quad \left. + \frac{n}{2}\theta''^2\right\} \uparrow \end{aligned}$$

in $\sum x_i$.

Example. If

$$f(x, \theta) = \exp\{p(\theta)K(x) + S(x) + q(\theta)\},$$

then

$$\frac{L(\theta')}{L(\theta'')} = \exp\{(p(\theta') - p(\theta'')) \sum K(x_i) + n(q(\theta') - q(\theta''))\}.$$

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- If $p(\cdot) \uparrow$, then

$$\frac{L(\theta')}{L(\theta'')} \uparrow$$

in

$$Y = \sum_{i=1}^n K(x_i).$$

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- If we test

$$H_0 : \theta = \theta', \quad H_1 : \theta < \theta'.$$

For any $\theta'' < \theta'$, we have

$$\frac{L(\theta')}{L(\theta'')} \leq k$$
$$\iff \sum_{i=1}^n K(x_i) \leq c.$$

This provides a UMPT.

- If we test

$$H_0 : \theta = \theta', \quad H_1 : \theta > \theta',$$

then

$$\left\{ \sum K(x_i) \geq c \right\}$$

is a UMPT, where c depends on α only.