Ch 9. Theory of Statistical Tests

9.1 Certain Best Tests

- How to construct good testing.

For simple hypothesis

$$H_0 : \theta = \theta', \qquad H_1 : \theta = \theta'',$$

where $\Theta = \{\theta', \theta''\}$

首都师范大学 数学科学学院 统计学系 Department of Statistics

1. Define the best test for

$$H_0 \leftrightarrow H_1.$$

2. NP Theorem

3. An example.

- Use rejection region to define a test.

  Given C, if $(x_1, \cdots, x_n) \in C$, then reject H_0, otherwise, accept H_0.

- The best rejection region $C \Longleftrightarrow$ the best test.

Def: The best rejection region C for the simple hypothesis test of level α:

For any subset A in the sample space satisfying

$$P((X_1, \cdots, X_n \in A)) = \alpha :$$

$$(a).\quad P\{(X_1,\cdots,X_n)\in C;H_0\}=\alpha$$

$$(b).\quad P\{(X_1,\cdots,X_n\in C;H_1)\}\geq$$

$$P\left((X_1,\cdots,X_n)\in A;H_1\right).$$

NP Theorem: $X_1, \cdots, X_n$ iid. $\sim f(x, \theta)$,

$$L(\theta) = \prod_{i=1}^{n} f(x_i, \theta),$$

where $\Theta = \{\theta', \theta''\}$. Let

$$C = \left\{ (x_1, \cdots, x_n) : \frac{L(\theta')}{L(\theta'')} \le k \right\}$$

$$C^* = \left\{(x_1, \cdots, x_n) : \frac{L(\theta')}{L(\theta'')} > k\right\}$$

is a complement of C.

$\alpha = P\{(X_1, \cdots, X_n) \in C; H_0\}$.

Then C is a best rejection region of size α for testing the simple hypothesis $H_0 \leftrightarrow H_1$.

Proof:

$$0 \leq ? \int_C L(\theta^{''}) - \int_A L(\theta^{''})$$

$$= \int_{C \cap A^*} L(\theta^{''}) - \int_{A \cap C^*} L(\theta^{''})$$

$$\geq \frac{1}{k} \int_{C \cap A^*} L(\theta^{'}) - \frac{1}{k} \int_{A \cap C^*} L(\theta^{'})$$

$$= \frac{1}{k}\left[\int_C L(\theta') - \int_A L(\theta')\right]$$

$$= \frac{1}{k}(\alpha - \alpha) = 0.$$

- **Example**: $X_1, \cdots, X_n$ iid. $\sim N(\theta, 1)$,

and

$$H_0 : \theta' = 0,\ H_1 : \theta'' = 1.$$

$$\frac{L(\theta')}{L(\theta'')} = \exp\{-\sum_{i=1}^{n} X_i + \frac{n}{2}\} \leq k$$

$$\Leftrightarrow \frac{1}{n}\sum X_i \geq \frac{1}{2} - \frac{1}{n}\ln k =: c$$

Take

$$C = \{(x_1, \cdots, x_n) : \frac{1}{n}\sum_{i=1}^{n} x_i \geq c\}.$$

If $\alpha = 0.05$, then $c = 1.645/\sqrt{n}$.

$$P(C; H_0) = \alpha,$$

$$P(C; H_1) = P(\bar{X} - 1 \geq c - 1; H_1)$$

$$= 1 - \Phi(\sqrt{n}(c - 1)).$$

If $n = 25$, then

$$P(C; H_1) = 1 - \Phi(1.645 - 5)$$
$$= \Phi(3.355) = 0.999.$$

Generalization: H_0 and H_1 are simple hypothesis and $X_1, \cdots, X_n$ iid.

$$L_0 = g(x_1, \cdots, x_n)$$

is joint pdf. of $X_1, \cdots, X_n$,

$$L_1 = h(x_1, \cdots, x_n)$$

is joint pdf. of $X_1, \cdots, X_n$.

Then the critical region for $H_0 \leftrightarrow H_1$ is

$$C = \{(x_1, \cdots, x_n) : \frac{L_0}{L_1} \le k\}$$

for some $k > 0$ with

$$\alpha = P((x_1, \cdots, x_n) \in C; H_0).$$

9.2 Uniformly Most Powerful Tests (UMPT)

$$H_0 \leftrightarrow H_1$$

Def.: C is a UMPT of α for the simple hypothesis $H_0 \leftrightarrow$ composite hypothesis H_1 $\Leftrightarrow$ C is a best of α for test $H_0 \leftrightarrow$ each simple hypothesis in H_1.

In general UMPT doesn't exist. If it exists, then NP theorem is used.

Example 2. Let $X_1, \cdots, X_n$ iid. $\sim N(0, \theta)$

$$H_0 : \theta = \theta' \leftrightarrow H_1 : \theta > \theta'$$

where $\Theta = \{\theta : \theta \geq \theta'\}$, the joint pdf. of $X_1, \cdots, X_n$ is

$$L(\theta) = \left(\frac{1}{2\pi\theta}\right)^{\frac{n}{2}} \exp\{-\frac{\sum_{i=1}^n x_i^2}{2\theta}\}.$$

For any $\theta'' > \theta'$, $k > 0$,

$$\frac{L(\theta')}{L(\theta'')} \le k$$

$$\Leftrightarrow \left(\frac{\theta''}{\theta'}\right)^{\frac{n}{2}} \exp\{-\frac{\theta''-\theta'}{2\theta'\theta''}\sum x_i^2\} \le k$$

$$\Leftrightarrow \sum x_i^2 \ge \frac{2\theta'\theta''}{\theta''-\theta'}[\frac{n}{2}\ln(\frac{\theta''}{\theta'}) - \ln k]$$

$$=: c.$$

Therefore

$$C = \{(x_1, \cdots, x_n) : \sum x_i^2 \geq c\}$$

is the best for

$$H_0 \leftrightarrow H_1 : \theta = \theta'',$$

where c is determined by α.

$$P\left(\frac{\sum X_i^2}{\theta'} \geq \frac{c_{H_0}}{\theta''}\right) = \alpha$$

$$\Rightarrow c = \theta' \chi_n^2(\alpha).$$

For $H_1 : \theta = \theta'''$, then

$$C = \{(x_1, \cdots, x_n) : \sum x_i^2 \geq c\}$$

is the same and also the best, C is UMPT.

If $n = 5, \alpha = 0.05, \theta' = 3,$

$$H_0 : \theta = 3, \qquad H_1 : \theta > 3,$$

then

$$c = 3 \times \chi^2_{15}(0.05) = 3 \times 25 = 75.$$

• Let $X_1, \cdots, X_n$ iid. $\sim f(x, \theta)$ ($\theta \in \Theta$).

Suppose that

$$Y = u(x_1, \cdots, x_n)$$

is a sufficient, then

$$L(\theta) =$$

$$k_1[u(x_1, \cdots, x_n); \theta] k_2(x_1, \cdots, x_n),$$

$\Rightarrow$

$$\frac{L(\theta^{'})}{L(\theta^{''})} = \frac{k_1(u(x_1, \cdots, x_n); \theta^{'})}{k_1(u(x_1, \cdots, x_n); \theta^{''})}$$

$$= \frac{k_1(y, \theta^{'})}{k_1(y, \theta^{''})}.$$

- A best test or UMPT depends on $u(x_1, \cdots, x_n)$ which is sufficient.

- If $\frac{k_1(y,\theta')}{k_1(y,\theta'')}$ is an increasing function of y for $\theta'' < \theta'$, then

$$\frac{L(\theta')}{L(\theta'')}$$

is called monotone likelihood ratio in the statistic $Y = u(x_1, \cdots, x_n)$.

Example. $X_1, \cdots, X_n$ **iid.** $\sim N(\theta, 1)$, **if** $\theta'' < \theta'$,

$$\frac{L(\theta')}{L(\theta'')} = \frac{\prod_{i=1}^{n} \exp\{-\frac{1}{2}(x_i - \theta')^2\}}{\prod_{i=1}^{n} \exp\{-\frac{1}{2}(x_i - \theta)^2\}}$$

$$= \exp\{(\theta' - \theta'') \sum x_i - \frac{n}{2}\theta'^2$$

$$+\frac{n}{2}\theta''^2\} \uparrow$$

in $\sum x_i$.

Example. If

$$f(x,\theta) = \exp\{p(\theta)K(x) + S(x) + q(\theta)\},$$

then

$$\frac{L(\theta')}{L(\theta'')} = \exp\{(p(\theta') - p(\theta''))\sum K(x_i) + n(q(\theta') - q(\theta''))\}.$$

Home Page
Title Page
Page 27 of 100
Go Back
Full Screen
Close
Quit

- If $p(\cdot)\uparrow$, then

$$\frac{L(\theta')}{L(\theta'')}\uparrow$$

in

$$Y=\sum_{i=1}^{n}K(x_i).$$

- If we test

$$H_0 : \theta = \theta^{'}, \quad H_1 : \theta < \theta^{'}.$$

For any $\theta'' < \theta^{'}$, we have

$$\frac{L(\theta^{'})}{L(\theta'')} \leq k$$

$$\Longleftrightarrow \sum_{i=1}^{n} K(x_i) \leq c.$$

This provides a UMPT.

- If we test

$$H_0 : \theta = \theta^{'}, \qquad H_1 : \theta > \theta^{'},$$

then

$$\{\sum K(x_i) \geq c\}$$

is a UMPT, where c depends on α only.