

## 8.2 Fisher Information CR Inequality

- $X \sim f(x, \theta), \theta \in \Theta = (a, b)$ .
- Regular cases:

$$f(x, \theta) > 0, \quad \frac{\partial}{\partial \theta} \int f = \int \frac{\partial}{\partial \theta} f,$$

Since  $\int f(x, \theta) = 1$ , then

$$\int \frac{\partial f(x, \theta)}{\partial \theta} dx = 0,$$

i.e.

$$\int \frac{\partial \ln f(x, \theta)}{\partial \theta} f(x, \theta) dx = 0.$$

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$$\Rightarrow \int \left[ \frac{\partial^2 \ln f(x, \theta)}{\partial \theta^2} f(x, \theta) + \frac{\partial \ln f(x, \theta)}{\partial \theta} \frac{\partial f(x, \theta)}{\partial \theta} \right] dx = 0,$$

$$\Rightarrow \int \left( \frac{\partial \ln f(x, \theta)}{\partial \theta} \right)^2 f(x, \theta) dx + \int \frac{\partial^2 \ln f(x, \theta)}{\partial \theta^2} f(x, \theta) dx = 0.$$

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Therefore,

$$\begin{aligned} I(\theta) &=:\int\left(\frac{\partial\ln f(x,\theta)}{\partial\theta}\right)^2f(x,\theta)dx \\ &=E\left[\frac{\partial\ln f(x,\theta)}{\partial\theta}\right]^2 \\ &=-\int\frac{\partial^2\ln f(x,\theta)}{\partial\theta^2}f(x,\theta)dx \\ &=-E\frac{\partial^2\ln f(x,\theta)}{\partial\theta^2}. \end{aligned}$$

- The larger of  $I(\theta)$ , the more information of  $\theta$ .
- If  $I(\theta) = 0$ , then  $\frac{\partial \ln f}{\partial \theta} = 0$ . No information in  $\ln f(x, \theta)$ .
- $\frac{\partial \ln f(x, \theta)}{\partial \theta}$  is an important function.

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MLE  $\hat{\theta}$  satisfies:

$$\sum_{i=1}^n \frac{\partial \ln f(x_i, \hat{\theta})}{\partial \theta} = 0.$$

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Example 1:  $X \sim N(\theta, \sigma_0^2)$ ,

$$f(x, \theta) = \frac{1}{\sqrt{2\pi}\sigma_0} \exp\left\{-\frac{(x - \theta)^2}{2\sigma_0^2}\right\}.$$

Then

$$\ln f(x, \theta) = -\ln(\sqrt{2\pi}\sigma_0) - \frac{(x - \theta)^2}{2\sigma_0^2}.$$

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$\Rightarrow$

$$\frac{\partial \ln f(x, \theta)}{\partial \theta} = \frac{x - \theta}{\sigma_0^2},$$

$\Rightarrow$

$$\frac{\partial^2 \ln f(x, \theta)}{\partial \theta^2} = -\frac{1}{\sigma_0^2}.$$

Then

$$I(\theta) = \frac{1}{\sigma_0^2}.$$

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**Example 2:**  $X \sim B(1, \theta)$ , then

$$\begin{aligned} \ln f(x, \theta) &= x \ln \theta \\ &\quad + (1 - x) \ln(1 - \theta), \end{aligned}$$

$$\frac{\partial^2 \ln f(x, \theta)}{\partial \theta^2} = -\frac{x}{\theta^2} - \frac{(1 - x)}{(1 - \theta)^2}.$$

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Hence,

$$\begin{aligned} I(\theta) &= E \left[ \frac{x}{\theta^2} + \frac{1-x}{(1-\theta)^2} \right] \\ &= \frac{1}{\theta} + \frac{1}{1-\theta} = \frac{1}{\theta(1-\theta)}. \end{aligned}$$

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- Suppose  $X_1, \dots, X_n$  iid.  $\sim f(x, \theta)$ , then

$$L(\theta) = \prod_{i=1}^n f(x_i, \theta)$$

and

$$\ln L(\theta) = \sum_{i=1}^n \ln f(x_i, \theta).$$

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$$\frac{\partial \ln L(\theta)}{\partial \theta} = \sum_{i=1}^n \frac{\partial \ln f(x_i, \theta)}{\partial \theta}.$$

$$\begin{aligned} I_n(\theta) &= E\left[\frac{\partial \ln L(\theta)}{\partial \theta}\right]^2 \\ &= \sum_{i=1}^n E\left(\frac{\partial \ln f(x_i, \theta)}{\partial \theta}\right)^2 \\ &= nI(\theta). \end{aligned}$$

- Rao-Cramer inequality:

$$Y = u(X_1, \dots, X_n) \sim \theta,$$

$$E(Y) = k(\theta).$$

$$\Rightarrow k(\theta) = \int \prod f(x_i, \theta) dx_i$$

$$k'(\theta) = \int u(x_1, \dots, x_n) \left( \sum_{i=1}^n \frac{\partial \ln f(x_i, \theta)}{\partial \theta} \right) \prod f(x_i, \theta) dx_i.$$

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Let  $Z = \sum_{i=1}^n \frac{\partial \ln f(x_i, \theta)}{\partial \theta}$ , then

$$EZ = 0,$$

$$\begin{aligned} \text{Var}(Z) &= nE \left[ \frac{\partial \ln f(X, \theta)}{\partial \theta} \right]^2 \\ &= nI(\theta). \end{aligned}$$

Since  $k'(\theta) = E(YZ)$ , then

$$k'(\theta) \leq \sqrt{\text{Var}(Y)\text{Var}(Z)}$$

$\Rightarrow$

$$\begin{aligned}\text{Var}(Y) &\geq \frac{[k'(\theta)]^2}{\text{Var}(Z)} \\ &= \frac{[k'(\theta)]^2}{nI(\theta)}.\end{aligned}$$



- If  $k(\theta) = \theta$ , then  $Y$  is unbiased and

$$\text{Var}(Y) \geq \frac{1}{nI(\theta)}$$

CR-lower bound.

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**Example 1:**  $X_1, \dots, X_n$  iid.  $\sim$

$N(\theta, \sigma^2)$  with known  $\sigma^2$ . Then

$$\text{Var}(Y) \geq \frac{\sigma^2}{n}.$$

**Example 2:**  $X_1, \dots, X_n$  iid.  $\sim B(1, \theta)$ .

Then

$$\text{Var}(Y) \geq \frac{\theta(1 - \theta)}{n}.$$



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**Def 1:**  $Y$  is a efficient unbiased estimator of  $\theta \iff$  the variance of  $Y$  attains the CR lower bound.

**Def 2:** Efficiency

$$\frac{\text{The CR lower bound}}{\text{Var}(Y)}$$

is called the efficiency of unbiased estimator  $Y$ .

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**Example 3:**  $X_1, \dots, X_n$  iid.  $\sim N(\mu, \theta)$ ,  
 $0 < \theta < +\infty$  with known  $\mu$ .

$$E(S^2) = \theta.$$

and

$$\ln f(x, \theta) = -\frac{(x - \mu)^2}{2\theta} - \frac{\ln \theta}{2} + c.$$

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$$I(\theta) = -E \left[ \frac{\partial \ln^2 f(X, \theta)}{\partial \theta^2} \right]$$
$$= E \left[ \frac{(X - \mu)^2}{\theta^3} - \frac{1}{2\theta^2} \right] = \frac{1}{2\theta^2}.$$

and

$$\frac{1}{nI(\theta)} = \frac{2\theta^2}{n}.$$

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Since

$$\begin{aligned} \text{Var}(S^2) &= \frac{2(n-1)}{(n-1)^2} \theta^2 = \frac{2\theta^2}{n-1} \\ &> \frac{2\theta^2}{n}, \end{aligned}$$

then

$$\text{eff}(S^2) = \frac{n-1}{n}.$$

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Take  $S_0^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2$ , then

$$\text{eff}(S_0^2) = 1.$$

- Asymptotically efficient:

$$\text{eff}(Y_n) \rightarrow 1 \quad (n \rightarrow \infty).$$

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